Magnetohydrodynamic flow constructions with fundamental solutions

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In this paper steady flows of an incompressible, viscous, electrically conducting fluid are constructed from fundamental solutions of magnetohydrodynamics in which the applied magnetic field is parallel to the velocity at infinity. The flat plate and the sphere are considered as examples, and approximate solutions are presented for the limiting cases of large and small Reynolds and magnetic Reynolds numbers. The effects of currents in the body are also considered, and it is found that unless the magnetic Prandtl number is larger than unity, currents in the body have negligible effect on the flow.

1. Introduction

A certain class of magnetohydrodynamic (MHD) problems involving the flow of an incompressible electrically conducting fluid over a solid body, in the presence of an applied magnetic field parallel to the main flow, can be described approximately by a set of linear equations (Gourdine 1960). Simple MHD flows over solids can then be constructed by superposing fundamental solutions of these equations to satisfy appropriate boundary conditions. Examples of this technique are presented in this paper

Chester (1957) treats MHD flow over a sphere as a perturbation of the Stokes flow, while Ludford (1959) treats the same problem as a perturbation of the Oseen flow. Lary (1960) studies this class of problems assuming the fluid has zero viscosity and finds that when the Alfvén number α is larger than unity the body produces a forward wake due to the propagation of Alfvén disturbances upstream. Hasimoto (1959) finds a similar phenomenon is his study of the problem with a fluid of infinite electrical conductivity. Other investigators have also considered problems in this class: Gotoh (1960), Yosinobu (1960), Blerkom (1960), Greenspan & Carrier (1959), and Greenspan (1960). In this paper, the fundamental solution or singular body approach is employed. This is not a new technique (Blerkom 1960), but it has the advantage of providing physical insight into the problem, and it also allows easy generalization of the problem to include the effects of currents in the body.

2. Mathematical description of the problem

The following set of dimensionless linearized equations describes the induced velocity field **u** and the induced magnetic field **h** (Gourdine 1960):

$$\frac{\partial \mathbf{u}}{\partial x} - \frac{1}{Re}\nabla^2\mathbf{u} - \alpha^2\frac{\partial \mathbf{h}}{\partial x} = -\nabla\overline{p} + \mathbf{f}, \tag{1a}$$

$$\frac{\partial \mathbf{h}}{\partial x} - \frac{1}{Rm}\nabla^2\mathbf{h} - \frac{\partial \mathbf{u}}{\partial x} = \mathbf{g}, \tag{1b}$$

$$\nabla.\mathbf{u} = \nabla.\mathbf{h} = 0, \tag{1c}$$

where $\mathbf{f} = \delta(r)\,\mathbf{a}$ represents a singular velocity disturbance in the flow and $\mathbf{g} = \delta(r)\,\mathbf{b}$ represents a singular magnetic disturbance in the flow; here $\delta(r)$ is the Dirac delta function, r being measured radially from the origin, and **a** and **b** are unit vectors. The other symbols are defined as follows: Re is the Reynolds number, Rm is the magnetic Reynolds number, α is the Alfvén number, and $\overline{p}$ is the *net* pressure—the sum of hydrostatic and magnetic pressure.

The solutions of equations (1) are the fundamental solutions for this class of MHD problems, and it has been shown (Gourdine 1960) that they can be written in the following form

$$u_{ij}(x,y,z) = u^u_{ij} + u^h_{ij} = \sum_{n=0}^{2} (U^u_n + U^h_n)\,\Gamma_{ij}(\lambda_n x, y, z), \tag{2a}$$

$$h_{ij}(x,y,z) = h^u_{ij} + h^h_{ij} = \sum_{n=0}^{2} K_n(U^u_n + U^h_n)\,\Gamma_{ij}(\lambda_n x, y, z). \tag{2b}$$

The *net* pressure is proportional to the x-component of the zeroth mode of the velocity ($n = 0$):

$$\overline{p}_j(x,y,z) = \overline{p}^u_j + \overline{p}^h_j = -(1-\alpha^2)\,(U^u_0 + U^h_0)\,\Gamma_{xj}(\lambda_0 x, y, z). \tag{3}$$

The fundamental solutions u_{ij}, h_{ij} and $\overline{p}_j$, are the velocity and magnetic field disturbances in the i-direction and the net pressure, respectively, due to singular disturbances that are in the j-direction at the origin. The fundamental solutions consist of contributions from the velocity singularity (superscript u) and the magnetic singularity (superscript h).

The function $\Gamma_{ij}(\lambda_n x, y, z)$ represents the fundamental solution of the Oseen-type equations

$$\frac{\partial \mathbf{u}}{\partial x} - \frac{1}{\lambda_n}\nabla^2\mathbf{u} = \delta(r)\,\mathbf{j}, \tag{4a}$$

$$\nabla.\mathbf{u} = 0. \tag{4b}$$

Here λ_n has the following three values

$$\lambda_0 = 0, \tag{5a}$$

$$\lambda_1 = \tfrac{1}{2}Re[(1+Pm) + \{(1+Pm)^2 - 4Pm(1-\alpha^2)\}^{\frac{1}{2}}], \tag{5b}$$

$$\lambda_2 = \tfrac{1}{2}Re[(1+Pm) - \{(1+Pm)^2 - 4Pm(1-\alpha^2)\}^{\frac{1}{2}}], \tag{5c}$$

where $Pm \equiv \sigma\mu\nu$ is the so-called magnetic Prandtl number. These are plotted in figures 1 and 2.

The magnetic mode parameters K_n have the three values

$$K_0 = 1, \tag{6a}$$

$$K_1 = \tfrac{1}{2}\alpha^{-2}[(1-Pm)-\{(1-Pm)^2+4\alpha^2 Pm\}^{\frac{1}{2}}], \tag{6b}$$

$$K_2 = \tfrac{1}{2}\alpha^{-2}[(1-Pm)+\{(1-Pm)+4\alpha^2 Pm\}^{\frac{1}{2}}]. \tag{6c}$$

These are plotted in figures 3 and 4.

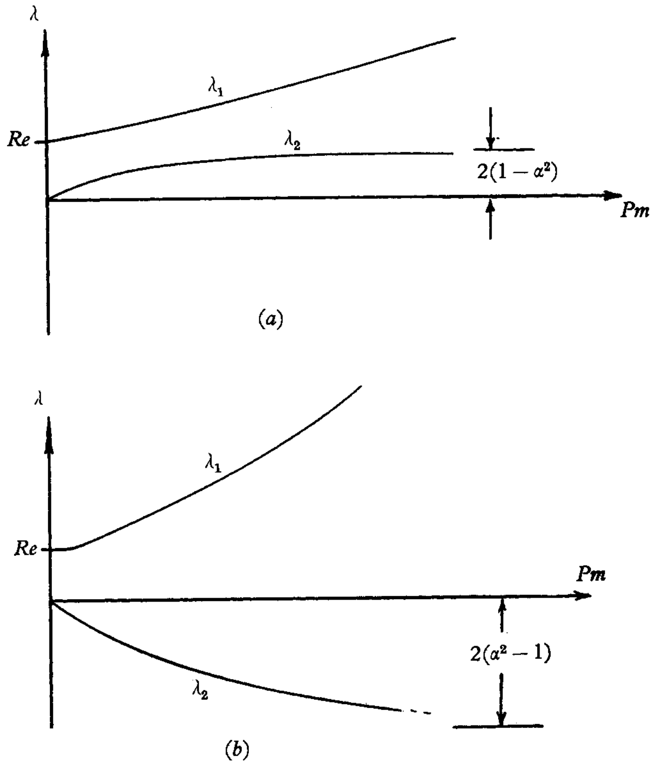

FIGURE 1. (*a*) $\alpha < 1$; (*b*) $\alpha > 1$.

The mode strengths in equations (2*a*) and (2*b*) consist of two parts; U_n^u is due to a velocity singularity and U_n^h is due to a magnetic singularity. A study of the equations (1) near the singularities at the origin yields the following expressions for the velocity singularity mode strengths,

$$U_0^u = 1, \tag{7a}$$

$$U_1^u = K_2/(K_2-K_1), \tag{7b}$$

$$U_2^u = -K_1/(K_2-K_1), \tag{7c}$$

and corresponding expressions for the magnetic singularity mode strengths,

$$U_0^h = 1, \tag{8a}$$

$$U_1^h = -Pm/(K_2-K_1), \tag{8b}$$

$$U_2^h = Pm/(K_2-K_1). \tag{8c}$$

These relations ensure that the fields are divergence free even at the origin.

MHD flow over a finite body may be described in terms of the fundamental solutions by the integral

$$u_i(x,y,z) = \iiint_{-\infty}^{\infty} u_{ij}^u(x,y,z;\,\xi,\eta,\zeta)\,S_j^u(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta + \iiint_{-\infty}^{\infty} u_{ij}^h(x,y,z,\xi,\eta,\zeta)\,S_j^h(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta, \quad (9)$$

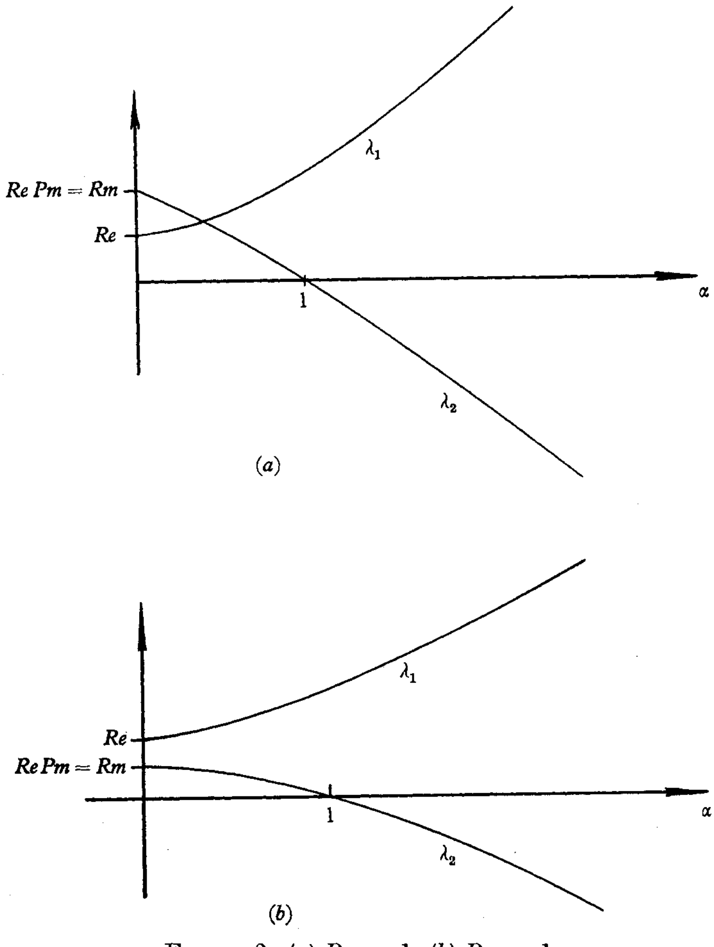

FIGURE 2. (*a*) $Pm > 1$; (*b*) $Pm < 1$.

where S_j^u and S_j^h are the strengths per unit volume of the distributed velocity and magnetic singularities respectively. A similar integral can be written for the magnetic field which, of course, also involves S_j^u and S_j^h. These two distribution functions can be determined, in principle at least, by satisfying two boundary conditions on the body. One boundary condition is that the total velocity must vanish at the body; therefore that $u_x = -1, u_y = 0, u_z = 0$ at the body. The other boundary condition is either on the magnetic field or the current density at the body. In the most general problem, the body carries an internal electric generator which establishes currents within itself.

The remainder of this paper concerns itself with the problem of solving for S_j^u and S_j^h. Only special cases are attempted because a general solution is probably impossible.

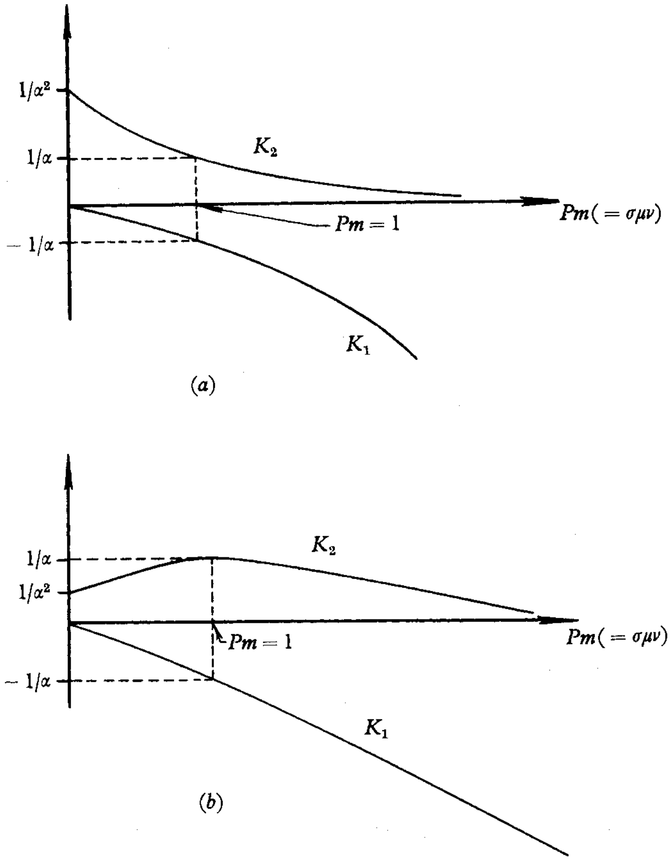

FIGURE 3. (a) $\alpha < 1$; (b) $\alpha > 1$.

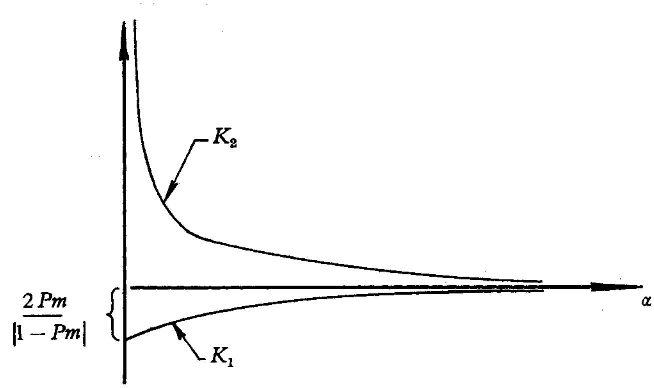

FIGURE 4

3. MHD drag of a finite flat plate

Following the usual procedure for constructing flows by superposition, let $S_x^u = f(\xi)$ be the strength of the velocity singularity in the range $\xi < x < \xi + d\xi$, and let $S_x^h = Mf(\xi)$ be the strength of the magnetic singularity in the same range. This is a special case in which the distribution of magnetic singularities is assumed to be proportional to the distribution of velocity singularities in order to demonstrate in a simple way the effects of currents in the body. The fundamental solution for this case can be written immediately in terms of Oseen's solution, that is

$$u_{ij} = U_1 \Gamma_{ij}(\lambda_1 x, y) + U_2 \Gamma_{ij}(\lambda_2 x, y), \tag{10}$$

where the mode strengths are given by

$$U_1 = U_1^u + MU_1^h, \tag{11a}$$

$$U_2 = U_2^u + MU_2^h. \tag{11b}$$

Oseen's solution for a singular flat plate with unit drag is

$$\Gamma_{i1} = U\left[\frac{1}{\lambda}\nabla(\chi - \phi) - \chi \mathbf{i}\right], \tag{12}$$

where $U = Re/\lambda\pi$ and $\lambda = Re$. The function $\phi(r) = \ln r$ is the potential of a two-dimensional source; it is the longitudinal part of the solution, and is chosen such that the flow is divergence-free near the origin. The Oseen function in two dimensions is $\chi = e^{\frac{1}{2}\lambda x} K_0(\frac{1}{2}\lambda r)$, where K_0 is the modified Bessel function of the second kind. At large distances from the origin, χ has the asymptotic behaviour $\chi \sim e^{-\frac{1}{2}\lambda(r-x)}/(\frac{1}{2}\lambda r)^{\frac{1}{2}}$. Near the origin $\chi \simeq \ln r$. Therefore χ behaves like a potential source near the origin; but for large positive x, it is essentially zero outside a parabolic region bounded by $y = c(x/\lambda)^{\frac{1}{2}}$. The strength of χ inside the parabola vanishes slowly, like $(1/x)^{\frac{1}{2}}$. However, for negative x, χ vanishes exponentially. Thus Oseen's solution contains a parabolic wake extending dowstream from the body.

According to equations (4) there are two wakes in this problem. If $\lambda_2 > 0$, there is a second wake downstream, and if $\lambda_2 < 0$, there is a second wake upstream. Equations (5) show that $\lambda_2 < 0$ if $\alpha > 1$, and $\lambda_2 > 0$ if $\alpha < 1$.

The distribution function $f(\xi)$ is such that the x-component of velocity on the plate vanishes; that is,

$$u_x(x, 0) = -1 = \int_{-1}^{1} \{U_1 \Gamma_{11}[\lambda_1(x-\xi), 0] + U_2 \Gamma_{11}[\lambda_2(x-\xi), 0]\} f(\xi)\, d\xi. \tag{13}$$

On the plate, the argument $\lambda(x-\xi) \leqslant \lambda < 1$, therefore

$$\Gamma_{11}[\lambda(x-\xi), 0] \simeq (Re/2\pi)(1 - \ln \tfrac{1}{4}\gamma_0 \lambda |x-\xi|), \tag{14}$$

where $\gamma_0 = e^\gamma$ and $\gamma = 0{\cdot}577$ (Euler's constant). Substitution of this into equation (13) yields

$$-1 = \int_{-1}^{1} \{U_1(C_1 - \ln|x-\xi|) + U_2(C_2 - \ln|x-\xi|)\} f(\xi)\, d\xi, \tag{15}$$

where

$$C_1 = 1 - \ln(\tfrac{1}{4}\gamma_0 \lambda_1), \tag{16a}$$

$$C_2 = 1 - \ln(\tfrac{1}{4}\gamma_0 \lambda_2). \tag{16b}$$

The solution of equation (15) is

$$f(\xi) = \frac{2/Re}{\{U_1(C_1+\ln 2)+U_2(C_2+\ln 2)\}}\frac{1}{(1-\xi^2)^{\frac{1}{2}}}. \tag{17}$$

The drag coefficient per unit length is therefore

$$C_D \equiv \int_{-1}^{1} f(\xi)\,d\xi = \frac{2\pi}{Re}\frac{1}{\{U_1(C_1+\ln 2)+U_2(C_2+\ln 2)\}}. \tag{18}$$

In the limit of zero magnetic interaction C_D approaches the classical hydrodynamic value

$$C_{D_0} = \frac{2\pi}{Re}\frac{1}{\{1-\ln(\gamma_0 Re/8)\}}, \tag{19}$$

where $Re = U_\infty a/\nu$.

Now consider the effect of the magnetic singularities on the mode strengths U_1 and U_2:

$$U_1 = \frac{K_2 - MPm}{K_2 - K_1}, \tag{20a}$$

$$U_2 = \frac{-K_1 + MPm}{K_2 - K_1}. \tag{20b}$$

The effect of a positive M is to decrease U_1 and increase U_2, and vice-versa for a negative M. In fact if $MPm = K_2$ the first mode is cancelled out entirely, and if $MPm = K_1$, the second mode is cancelled out. In the former case, with only the second mode remaining,

$$C_D \simeq \frac{2\pi}{Re}\frac{1}{C_2+\ln 2} = \frac{2\pi}{Re}\frac{1}{1-\ln(\gamma_0 Rm/8)}, \tag{21}$$

and in the latter case, with only the first mode remaining,

$$C_D \simeq \frac{2\pi}{Re}\frac{1}{C_1+\ln 2} = \frac{2\pi}{Re}\frac{1}{1-\ln\gamma_0 Re/8} = C_{D_0}. \tag{22}$$

Notice that the effect of currents in the body is proportional to the magnetic Prandtl number Pm, which is usually a very small number.

It is also possible to solve equation (13) for $f(\xi)$ when λ_1 and λ_2 are large, because in the limit $\lambda \to \infty$,

$$\Gamma_{11}[\lambda x, 0] \sim -\frac{Re}{2\pi}\left(\frac{\pi}{\lambda|x|}\right)^{\frac{1}{2}}, \quad (x \neq 0). \tag{23}$$

Furthermore, for $\alpha > 1$ (strong magnetic interaction), the first mode only produces disturbances downstream of the singularity, while the second mode only produces disturbances upstream of the singularity; therefore the integral equation for $f(\xi)$ becomes

$$-1 = -\frac{Re}{2\pi}\int_0^x U_1\left(\frac{\pi}{\lambda_1(x-\xi)}\right)^{\frac{1}{2}} f(\xi)\,d\xi - \frac{Re}{2\pi}\int_x^1 U_2\left(\frac{\pi}{\lambda_2(\xi-x)}\right)^{\frac{1}{2}} f(\xi)\,d\xi. \tag{24}$$

For $\alpha < 1$ (weak magnetic interaction), the first mode and the second mode only create disturbances downstream of the singularity; therefore,

$$-1 = -\frac{Re}{2\pi}[U_1(\pi/\lambda_1)^{\frac{1}{2}} + U_2(\pi/\lambda_2)^{\frac{1}{2}}]\int_0^x \frac{f(\xi)\,d\xi}{(x-\xi)^{\frac{1}{2}}} \quad (\alpha < 1). \tag{25}$$

This is Abel's integral equation, and has the solution

$$f(\xi) = \frac{1}{\sqrt{\xi}}\frac{2}{Re}\frac{1}{U_1(\pi/\lambda_1)^{\frac{1}{2}} + U_2(\pi/\lambda_2)^{\frac{1}{2}}}, \quad (\alpha < 1). \tag{26}$$

The drag coefficient is

$$C_D \equiv \int_0^1 f(\xi)\,d\xi = \frac{4}{Re\sqrt{\pi}}\frac{1}{U_1\lambda_1^{-\frac{1}{2}}+U_2\lambda_2^{-\frac{1}{2}}}. \tag{27}$$

Notice that as α or Pm tends to zero C_D approaches the classical value for high Reynolds number flow

$$C_{D_0} \sim 4/(\pi Re)^{\frac{1}{2}}. \tag{28}$$

For $\alpha > 1$, finding $f(\xi)$ from equation (24) is somewhat more difficult; however, Greenspan (1960) provides the solution. In the special cases in which either U_1 or U_2 vanishes, equation (24) is again easy to solve. If U_2 vanishes we again have Abel's integral equation, and the solution

$$f(\xi) = \frac{1}{\sqrt{\xi}}\frac{2}{Re}\frac{1}{U_1(\pi/\lambda_1)^{\frac{1}{2}}}. \tag{29}$$

If U_1 vanishes we have an equation which can be converted into Abel's equation by the substitution $\xi = 1-\eta$, and the resulting solution is

$$f(\xi) = \frac{1}{(1-\xi)^{\frac{1}{2}}}\frac{2}{Re}\frac{1}{U_2(\pi/\lambda_1)^{\frac{1}{2}}}. \tag{30}$$

Thus, when the second mode vanishes ($U_2 = 0$), the leading edge is singular and there is a trailing wake, but when the first mode vanishes ($U_1 = 0$) the trailing edge is singular and there is a leading wake.

For a flat plate without internal currents, U_2 vanishes if the electrical conductivity is zero, and U_1 vanishes if the electrical conductivity is infinite. It is also possible to make U_2 or U_1 vanish by properly distributing the currents in the flat plate as suggested by equations (20*a*) and (20*b*). Thus, by making $MPm = K_2$ we can make U_1 vanish, and by making $MPm = K_1$, we can make U_2 vanish. The current density generated in the plate, and the vorticity, must be at right angles to the flow and in opposite directions on the top and bottom sides.

The results presented in this section are in agreement with the results of Greenspan (1960). Greenspan uses an entirely different mathematical technique to arrive at equation (24), and does not concern himself with currents generated in the flat plate.

4. MHD drag of a sphere

In ordinary hydrodynamics, low Reynolds number flow over a sphere is constructed approximately by superposing a uniform flow, the flow of a singular needle, and the flow of a potential dipole; therefore it is reasonable to expect that low MHD Reynolds number flow over a sphere can be approximated by replacing the singular needle by a singular MHD needle as follows:

$$\begin{aligned}\mathbf{U} = \mathbf{i} &+ U_1^u\left\{-\frac{1}{\lambda_1}\nabla\left[\frac{e^{-\frac{1}{2}\lambda_1(r-x)}-1}{r}\right]+\frac{e^{-\frac{1}{2}\lambda_1(r-x)}}{r}\mathbf{i}\right\}\\ &+ U_2^u\left\{-\frac{1}{\lambda_2}\nabla\left[\frac{e^{-\frac{1}{2}\lambda_2(r-x)}-1}{r}\right]+\frac{e^{-\frac{1}{2}\lambda_2(r-x)}}{r}\mathbf{i}\right\}\\ &+ U_0^u\left\{\nabla\left[\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right]\right\},\end{aligned} \tag{31}$$

where the ratio of mode strengths for the singular MHD needle is

$$U_2^u/U_1^u = -K_1/K_2. \tag{32}$$

The mode strengths U_1^u and U_0^u are then determined by expanding the exponentials in equation (31) while satisfying the boundary condition on the sphere which demands that $U = 0$; they are

$$U_1^u \simeq -\frac{3}{2}\frac{K_1}{K_2-K_1}\frac{1}{1-\frac{3}{8}(K_2\lambda_2-K_1\lambda_1)/(K_2-K_1)}, \tag{33a}$$

$$U_0^u \simeq \frac{1}{4}\frac{1}{1-\frac{3}{8}(K_2\lambda_2-K_1\lambda_1)/(K_2-K_1)}, \tag{33b}$$

where, from equations (5) and (6)

$$(K_2\lambda_2-K_1\lambda_1)/(K_2-K_1) = Re. \tag{33c}$$

The drag of the sphere is due to a singular MHD needle of strength

$$\tfrac{3}{2}(4\pi/Re)\,(1+\tfrac{3}{8}Re),$$

therefore the drag is

$$D = 6\pi\rho\nu U_\infty a(1+\tfrac{3}{8}Re). \tag{34}$$

This is precisely the same drag as is obtained without magnetic interaction. It appears that for $\alpha < 1$, up to terms of order λ, the magnetic interaction has no effect on drag.

It is easy to obtain the solution for $\alpha > 1$ simply by changing the sign of x in equation (31) to account for the upstream wake when $\lambda_2 < 0$. The result is

$$U_1^u \simeq -\frac{3}{2}\frac{K_1}{K_2-K_1}\frac{1}{1-\frac{3}{8}(-K_2\lambda_2-K_1\lambda_1)/(K_2-K_1)}, \tag{35a}$$

$$U_0^u \simeq \frac{1}{4}\frac{1}{1-\frac{3}{8}(-K_2\lambda_2-K_1\lambda_1)/(K_2-K_1)}, \tag{35b}$$

where

$$K \equiv \frac{K_2\lambda_2+K_1\lambda_1}{K_2-K_1} = \frac{2Ha^2+Re^2-ReRm}{\{(Re-Rm)^2+4Ha^2\}^{\frac{1}{2}}}, \tag{35c}$$

and the drag is

$$D = 6\pi\rho\nu U_\infty a(1+\tfrac{3}{8}K) \tag{36}$$

provided that Ha, Re, Rm are all less than unity. In the limit Re, $Rm \to 0$ with $a \to \infty$, such that $Ha^2 \equiv ReRma^2$ is finite but less than unity, Chester's (1957) result is obtained:

$$D = 6\pi\rho\nu U_\infty a(1+\tfrac{3}{8}Ha). \tag{37}$$

These results also agree with those of Ludford (1959).

5. Conclusions

The fundamental solution approach to this class of MHD flow problems is a convenient method of constructing solutions. With a knowledge of the properties of the MHD singularities involved it is a simple matter to deduce the physical properties of the solutions. Some of these solutions may eventually prove useful in the design of probes to measure certain physical properties of a flowing conductor.

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